$\mathscr{R}$-solver approach to the maximal regularity of the free boundary problem for the Navier Stokes equations

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honey

beer

tornado

airplane

MFM group in Czech

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I visited Math. Institue first time a long time ago, 199? to participate in a summer school of PDE. Since then, I have a lot of oppotunity to visit here, and I have met a lot of excellent mathematicians, and I was and am able to have valuable methematical exchanges. In this opportunity, I represent my sincere thank to

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Jiri Neustupa, Milan Pokorny, Stanislav Kracmar, Tomas Bodnar, Ondrej Kreml, Ivan Straskraba, Antonin Novotny.

ρ: mass density, $\mathbf{u}$ velocity.

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0 \quad \text{conservation of mass}$$
$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \operatorname{div} \mathbb{T}(\mathbf{u}, P) \quad \text{conservation of momentum}$$

Euler consider this equations with $\mathbb{T}(\mathbf{u}, P) = -P$ (Pressure)
ρ is constant, $\operatorname{div} \mathbf{u} = 0$ and

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla P.$$

Euler flow: Once a vortex is created, it does not disappear !

Navier-Stokes flow: Vortex is created and the it does disappear, and .... repeated the viscosity was found.

$$\rho(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) - \mu\Delta\mathbf{u} + \nabla P = 0, \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0.$$

Oseen, C. W.: Neuere Methoden und Ergebnisse in der Hydrodynamik. Leipzig, Akad. Verlagsgesellschaft M.B.H., 1927. (Engineer's Bible).

$\Longrightarrow$

R. Finn

$\Longrightarrow$

Š. Nečasová, S. Kracmar, J. Neustupa, P. Penel

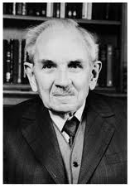
Mathematical theory of the fluid flow around a rotating and translating body

Weak solutions of Navier-Stokes equations:

$$\frac{\rho}{2}\int |\mathbf{u}(x,t)|^2\,dx + \mu\int_0^T\int |\nabla\mathbf{u}(x,t)|^2\,dtdx \leq \frac{\rho}{2}\int |\mathbf{u}(x,0)|^2\,dx.$$

•Jean Leray: 7 November 1906 – 10 November 1998)[1] was a French mathematician,
• Juliusz Paweł Schauder ; 21 September 1899, Lwów, Austria-Hungary – September 1943, Lwów, Occupied Poland) was a Polish mathematician.

• E. Hopf: Galerkin Method: Construct approximation solutions
Problem Prove the uniqueness of Leray-Hopf solutions



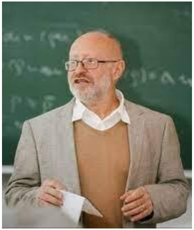
Jean Leray - Royal society (1991)

Weak solution of he Navier-Stokes equations describing the compressible viscous flows:

$$\partial_t \rho + \mathrm{div}\,(\rho \mathbf{u}) = 0 \quad \text{conservation of mass}$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \mathbf{u} = \mathrm{div}\,\mathbb{T}(\mathbf{u}, P) \quad \text{conservation of momentum}$$

$P = c\rho^{\gamma}$. Barotropic case
P. L. Lions, E. Feireisl, Š. Nečasová, Hana Petzeltova
Milan Pokorný, Ondřej Kreml, Antonín Novotný.



Problem: Prove the uniqueness of Leray-Hopf solutions

Unique exsitence of strong solutions

- Local in time unique existence of strong solutions without any size assumption of initial data
- Global in time unique existence of strong soluions for small initial data

The study of this topics started rather later than studying the weak solutions. And the study deeply depends on the theory of the linear PDE, Stokes equations.

Hille - Yoshida – Phillips – Miyadera theory.

$X \subset Y$ two Banach spaces, $A : X \to Y$ bounded linear operator

$$\partial_t u - Au = F \quad \text{for } t > 0, \quad u|_{t=0} = u_0.$$

Corresponding resolvent problem:

$$\lambda v - Av = f$$

has a unique solution $u \in X$ satisfying the resolvent estimate:

$$|\lambda| \|v\|_Y \leq C\|f\|_Y.$$

for every $\lambda \in \Sigma_\epsilon + \gamma$ with some $\epsilon \in (0, \pi/2)$ and $\gamma > 0$.
There exists a continuous analytic semigroup $\{T(t)\}_{t \geq 0}$ such that

$$u = T(t)u_0 + \int_0^t T(t-s)F(s)\,ds.$$

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \epsilon\}.$$

Fujita - Kato theory

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \mathbf{u}|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0$$

Fujita-Kato constructed a Stokes semigroup $\{T(t)\}_{t\geq 0}$ for the Stokes equations:

$$\partial_t \mathbf{u} - \Delta\mathbf{u} + \nabla\mathfrak{p} = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \mathbf{u}|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0$$

They wrote Navier-Stokes equations as

$$\mathbf{u} = T(t)\mathbf{u}_0 - \int_0^t T(t-s)(P\mathbf{u}\cdot\nabla\mathbf{u})(s)\,ds, \quad P : \text{Solenoidal Projection}$$

By using the Banach fixed point theorem, they proved the existence of unique strong solutions.

Free boundary problem for the NS eq.

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbb{T}(\mathbf{v}, \mathfrak{p}) = 0, \quad \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega_t,\ t > 0, \\ \mathbb{T}(\mathbf{v}, \mathfrak{p})\mathbf{n}_t = 0, \quad V_{\partial\Omega_t} = \mathbf{v}\cdot\mathbf{n}_t & \text{in } \partial\Omega_t,\ t > 0, \\ (\mathbf{v}, \Omega_t)|_{t=0} = (\mathbf{v}_0, \Omega). \end{cases} \tag{1}$$

- Ω_t a time dependent domain, $\partial\Omega_t$ its boundary, $\mathbf{n}_t$ unit outer normal to $\partial\Omega_t$,
- $\mathbf{v}$ the velocity field, $\mathfrak{p}$ the pressure field,
- $\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$: time derivative along the flow line,
- $\mathbb{T}(\mathbf{v}, \mathfrak{p}) = \mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbb{I}$ Cauchy stress tensor
- $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + \nabla\mathbf{v}^\top$ doubled deformation tensor,
- $\mu > 0$, constants describing the viscous coefficient and the coefficinet of the surface tension
- $V_{\partial\Omega_t}$ the evolution speed of $\partial\Omega_t$ in $\mathbf{n}_t$ direction,
- Ω_t, $\mathbf{v}$ and $\mathfrak{p}$ are unknown.
- $\mathbf{v}_0$ is an initial data and Ω is a reference domain in $\mathbb{R}^d$ $(d \geq 2)$.

Let $\phi_t : y \to x = \phi_t(y) : \Omega \to \Omega_t$ flow map for $t \geq 0$ with $\phi_t(y)|_{t=0} = y$. Let $\partial_t\phi_t(y) = \mathbf{u}(y, t)$ be the Lagrangian velocity field. And then, we have the Lagrange transformation:

$$x = \phi_t(y) = y + \int_0^t \mathbf{u}(y, s)\, ds$$

Assume that

$$\int_0^T \|\nabla\mathbf{u}(\cdot, s)\|_{L_\infty(\Omega)}\, ds << 1.$$

Then, the map $y \to x$ is invertible, and $\mathbf{v}(x, t) = \mathbf{u}(y, t)$.
Let $\mathbb{A} = \partial x/\partial y$ (Jacobi matrix), and then $\det\mathbb{A} = 1$ as follows from $\mathrm{div}_x\mathbf{v} = 0$. We have

$$\nabla_x\mathbf{v} = \mathbb{A}^\top\nabla_y\mathbf{u}, \quad \mathrm{div}_x\mathbf{v} = \mathbb{A}^\top : \nabla_y\mathbf{u} = \mathrm{div}_y(\mathbb{A}\mathbf{u}), \quad \mathbf{n}_t = \frac{\mathbb{A}^\top\mathbf{n}}{|\mathbb{A}^\top\mathbf{n}|}$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$.
Recall that for $d \times d$ matrices $\mathbb{A} = (A_{j,k})$ and $\mathbb{B} = (B_{j,k})$, we write $\mathbb{A} : \mathbb{B} = \sum_{j,k}^d A_{j,k}B_{j,k}$.

With the above notation, Problem (1) in Lagrangian coordinates reads

$$\begin{cases} \partial_t \mathbf{u} - \mathrm{div}\,(\mu \mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbb{I}) = \mathbf{F}(\mathbf{u}) & \text{in } \Omega \times \mathbb{R}_+, \\ \mathrm{div}\,\mathbf{u} = G_{\mathrm{div}}(\mathbf{u}) = \mathrm{div}\,\mathbf{G}(\mathbf{u}) & \text{in } \Omega \times \mathbb{R}_+, \\ (\mu \mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbb{I})\mathbf{n}_0 = \mathbf{H}(\mathbf{u})\mathbf{n}_0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ \mathbf{u}|_{t=0} = \mathbf{a} & \text{in } \Omega, \end{cases} \tag{2}$$

$$\begin{aligned} \mathfrak{q}(y,t) &:= \mathfrak{p}(x,t), \\ \mathbf{F}(\mathbf{u}) &:= (\mathbb{I} - (\mathbb{A}_{\mathbf{u}}^{\top})^{-1})\Big(\partial_t \mathbf{u} - \mu\Delta_y \mathbf{u}\Big) + \mu \mathbb{A}_{\mathbf{u}}^{-1}\mathrm{div}_{\,y}\Big((\mathbb{A}_{\mathbf{u}}\mathbb{A}_{\mathbf{u}}^{\top} - \mathbb{I})\nabla_y \mathbf{u}\Big) \\ &\quad + \mu\nabla_y\Big((\mathbb{A}_{\mathbf{u}}^{\top} - \mathbb{I}) : \nabla_y \mathbf{u}\Big), \\ G_{\mathrm{div}}(\mathbf{u}) &:= (\mathbb{I} - \mathbb{A}_{\mathbf{u}}^{\top}) : \nabla_y \mathbf{u}, \\ \mathbf{G}(\mathbf{u}) &:= (\mathbb{I} - \mathbb{A}_{\mathbf{u}})\mathbf{u}, \\ \mathbf{H}(\mathbf{u}) &:= \mu \mathbf{D}_y(\mathbf{u})(\mathbb{I} - \mathbb{A}_{\mathbf{u}}^{\top}) + \mu\Big\{(\nabla_y \mathbf{u})^{\top}(\mathbb{I} - \mathbb{A}_{\mathbf{u}}) + \Big(\mathbb{I} - (\mathbb{A}_{\mathbf{u}}^{-1})^{\top}\Big)(\nabla_y \mathbf{u})^{\top}\mathbb{A}_{\mathbf{u}}\Big\}\mathbb{A}_{\mathbf{u}}^{\top}. \end{aligned} \tag{3}$$

Idea of Maximal Regularity Theorem

Let X, Y, Z be three Banach spaces and $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(X, Z) \cap \mathcal{L}(Z, Y)$. Let me consider the time dependent problem:

$$\partial_t u - Au = f \ \text{ in } \Omega \times \mathbb{R}, \quad Bu = g \ \text{ on } \partial\Omega \times \mathbb{R}.$$

Applying Laplace-Fourier transform:

$$\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t)\, dt = \int_{\mathbb{R}} e^{-i\tau t}(e^{-\gamma t} f(t))\, dt \quad (\lambda = \gamma + i\tau).$$

noting that $\mathcal{L}[\partial_t \mathbf{u}] = \lambda \mathcal{L}[\mathbf{u}]$,

$$\lambda \hat{u} - A\hat{u} = \hat{f} \ \text{ in } \Omega, \quad B\hat{u}|_{\partial\Omega} = \hat{g}.$$

Here $\hat{f} = \mathcal{L}[f]$.

The parameter λ runs through $\Sigma_\epsilon + \gamma$ with $\epsilon \in (0, \pi/2)$ and $\gamma > 0$, where

$$\Sigma_\epsilon = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon\}.$$

Construct a solution operator: $\mathcal{S}(\lambda) : Y \times Y \times Z \to X$ such that

$$\hat{u} = \mathcal{S}(\lambda)(\hat{f}, \lambda^\alpha \hat{g}, \hat{g})$$

And then, u is obtained by

$$u = \mathcal{L}^{-1}[\mathcal{S}(\lambda)(\hat{f}, \lambda^\alpha \hat{g}, \hat{g})].$$

Her, $\mathcal{L}^{-1}$ denotes the Laplace-Fourier inverse transform defined by

$$\mathcal{L}^{-1}[f](t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{\lambda t} f(\tau)\, d\tau = e^{\gamma t} \int_{\mathbb{R}} e^{i\tau t} f(\tau)\, d\tau \quad \lambda = \gamma + i\tau.$$

If we introduce the operator $\Lambda^\alpha g = \mathcal{L}^{-1}[\lambda^\alpha \mathcal{L}[f]]$, we can write

$$u = \mathcal{L}^{-1}[\mathcal{S}(\lambda)\mathcal{L}[(f, \Lambda^\alpha g, g)]] = e^{\gamma t}\mathcal{F}^{-1}[\lambda^\alpha \mathcal{F}[e^{-\gamma t}(f, \Lambda^\alpha g, g)]].$$

This arguments has been done in the L_2 framework in 1970's by
Agranovich-Visik: parameter elliptic equations and parabolic equations,
R. Sakamoto: mixed problem for the strictly hyperbolic equatuions.

Operator valued Fourier multiplier: For $m(\xi) \in L^\infty(\mathbb{R}^N, \mathcal{L}(E,F))$, we set

$$T_m f = \mathcal{F}_\xi^{-1}[m(\xi)\mathcal{F}[f](\xi)] \quad f \in \mathcal{S}(\mathbb{R}, E)$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote respective Fourier transformation and inverse Fourier transformation, which are defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx \quad \mathcal{F}^{-1}[f](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} f(\xi)\,d\xi.$$

L. Weis, *Operator-valued Fourier multiplier theorems and maximal L^p-regularity*, Math. Ann., 319(4)(2001), 735–758. DOI:10.1007/PL00004457.

Theorem (Weis operator valued Fourier multiplier theorem)

Let E and F be two UMD Banach spaces. Let $m(\tau) \in C^1(\mathbb{R}, \mathcal{L}(E,F))$ and assume

$$\mathcal{R}_{\mathcal{L}(E,F)}\{m(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\} \leq r_b,$$
$$\mathcal{R}_{\mathcal{L}(E,F)}\{\tau m'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\} \leq r_b$$

with some constant $r_b > 0$. Then, for any $p \in (1,\infty)$, $T_m \in \mathcal{L}(L_p(\mathbb{R},E), L_p(\mathbb{R},F))$ and

$$\|T_m f\|_{L_p(\mathbb{R},F)} \leq C_p r_b \|f\|_{L_p(\mathbb{R},E)}$$

with some constant C_p depending solely on p.

Thanks to Weis's theorem, if the operator $\mathcal{S}(\lambda)$ is $\mathcal{R}$ bounded, then we can show the existence of solutions of time dependent problem:

$$\partial_t u - Au = f, \quad Bu = g \quad \text{for } t \in \mathbb{R}$$

which possess the maximal regularity estimate:

$$\int_{\mathbb{R}} (e^{-\gamma t}\|\partial_t u(\cdot, t)\|_Y)^p \, dt + \int_{\mathbb{R}} (e^{-\gamma t}\|u(\cdot, t)\|_X)^p \, dt$$
$$\leq C(\int_{\mathbb{R}} e^{-\gamma t}\|f\|_Y)^p \, dt + \int_{\mathbb{R}} (e^{-\gamma t}\|g(\cdot, t)\|_Z \, dt + \int_{\mathbb{R}} (e^{-\gamma t}\|\Lambda^\alpha g(\cdot, t)\|_Y \, dt.$$

For the initial value problem:

$$\partial_t u - Au = 0, \ \ Bu = 0 \quad \text{for } t > 0, \quad u|_{t=0} = u_0$$

We can show the existence of C_0 analytic semigroup $\{T(t)\}_{t\geq 0}$.
Thus, we have the maximal L_p regularity. Namely, pde

$$\partial_t u - Au = f, \quad Bu = g \quad \text{for } t > 0, \quad u|_{t=0} = u_0$$

is a one to one onto map from $\mathcal{X}_p$ onto $L_p((0, \infty), X) \cap W^1_p((0, \infty), Y)$. Here,

$$\mathcal{X}_p = L_p((0, \infty), Y) \times W^\alpha_p((0, \infty), Y) \times L_p((0, \infty), Z) \times (Y, X)_{1-1/p,p}.$$